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A Study on Graph Coloring

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Abstract-The Swiss Mathematician Leonhard Euler is considered as the Father of Graph theory. Today graph theory has matured into a full-fledged theory from a mere collection of challenging games and interesting puzzles. Peculiarity of Graph theory is that it depends very little on other branches of Mathematics and is independent in itself. Graph coloring enjoys many practical applications as well as theoretical challenges. Graph coloring is still a very active field of research. This paper consists of III Sections. Section I involves Introduction to Graph theory and Introduction to Graph Coloring. Section II is Vertex Coloring and Upper Bounds: in which Chromatic Polynomials and Chromatic Partitioning, Properties of Chromatic Numbers, Color Class, some important Theorems, Propositions, are discussed. In Section III Edge Coloring, Enumerative Aspects are discussed.

Index Terms- Coloring of a Graph, Chromatic Polynomials, Chromatic Number, Edge Coloring, Vertex Coloring, Upper Bounds and Coloring of planar graphs.

Section I: Introduction

1.1GraphTheory:

Graph theory is widely regarded as the most delightful branch of mathematics. This is because of its twin nature; it contains the cleverest proofs in all the abstract reasoning and it has the most comprehensive range of applicability to any contemporary science**.** Today graph theory has matured into a full-fledged theory from a mere collection of challenging games and interesting puzzles**.** Peculiarity of Graph theory is that it depends very little on other branches of Mathematics and is independent in itself.

Many Mathematicians have contributed to the growth of this `theory. **EULER** (1707-1782) became the father of graph theory when he settled a famous unsolved problem of his days called the **Konigsberg Bridge Problem**.

The Konigsberg bridge problem is regarded as the first paper in the history of graph theory. Two islands C and D, formed by the pregel river in Konigsberg were connected to each other and to the banks A and B with seven bridges. The Konigsberg bridge problem asks if it is possible to find a walk through the city of Konigsberg (now Kaliningrad, Russia) in such a way that we cross every bridge exactly once. Euler observed that the choice of route inside each land mass is irrelevant and thus the problem can be modeled in abstract terms by representing land masses with points (or capital letters as in Euler's original solution) and bridges between them with links between pairs of points. Such an abstract description of the problem naturally leads to the notion of a graph.

Definitions and Notations:

Graph: A graph G = (V, E) consists of an arbitrary set of objects V called vertices and a set E which contains unordered pairs of distinct elements of V called edges.

Adjacent: Two vertices in a graph are adjacent if there is an edge containing both of them. Two edges are adjacent if they contain a common vertex. Adjacent vertices are called neighbors.

Degree: For any vertex v in a graph, the degree of the vertex is equal to the number of edges which contain the vertex. The degree of v is denoted by d(v).

Regular Graph: A graph in which every vertex has the same degree is called a regular graph. If all vertices have degree k, the graph is said to be k-regular.

Complete Graph: The complete graph on n vertices K_n consists of the vertex set V = {v1,v2,…,vn} and the edge set E containing all pairs (vi,vj) of vertices in V.

A **bipartite graph** (bi – graph) G is a graph whose vertex set V(G) can be partitioned into subsets V_1 and V_2 such that every edge in G joins a vertex in V_1 to a vertex in V_2 . If G contains every edge joining a vertex of V1 to every vertex of V_2, then G is called a **complete bipartite graph**. If V_1 andV_2 have m and n vertices we write G = $K_{m, n}$.

A **star** is complete bigraph $K_{1, n}$.

Isomorphic: Two graphs are isomorphic if there exists a one-to-one correspondence between their vertex sets (i.e. a re-labeling) which induces a one-to-one correspondence between their edge sets. More formally, if L is a re-labeling which maps the vertices of G to the vertices of H, then the edge set of H is precisely the set of edges (L(v),L(w)) where (v,w) is an edge in G.

Sub-graph: A graph G1 = (V1, E1) is a sub-graph of G2 = (V2, E2) whenever V1 ⊆ V2 and E1 ⊆ E2.

Path: A path of length n is the graph Pn on n+1 vertices {v0, v1, v2, …, vn} with n edges (v0,v1), (v1,v2), …, (vn-1,vn).

Cycle: A cycle of length n is the graph C_n on n vertices {v0, v1, v2,…, vn-1} with n edges (v0, v1), (v1, v2), …, (vn-1,v0). We say that a given graph contains a path (or cycle) of length n if it contains a sub-graph which is isomorphic to P_n (or C_n).

Connected: A graph that contains a path between every pair of vertices is connected. Every graph consists of one or more disjoint connected sub-graphs called the connected components.

Distance: The distance between two connected vertices is the length of the shortest path between the vertices.

Diameter: The diameter of a connected graph is the maximum distance between any two vertices in the graph.

Forests and Trees: A graph which does not contain a cycle is called a forest. If it is a connected graph, it is called a tree. The connected components of a forest are trees.

End-points and Isolated Vertices: An end-point is a vertex with degree 1. An isolated vertex is a vertex with degree 0.

Hamiltonian Graph: A graph which contains a Hamiltonian cycle, i.e. a cycle which includes all the vertices, is said to be Hamiltonian.

Walks, Trails, and Circuits: A walk in a graph is a sequence of adjacent edges. A trail is a walk with distinct edges. A circuit is a trail in which the first and last edge are adjacent.

Eulerian Graph: A trail which includes all of the edges of a graph and visits every vertex is called an Eulerian Tour. If a graph contains an Eulerian tour which is a circuit, i.e. an Eulerian circuit, the graph is simply said to be Eulerian.

A graph is **acyclic** if it has no cycles. A tree is a connected acyclic graph.

A **clique** is a subset of vertices of an undirected graph such that its induced subgraph is complete. The **clique** graph K (G) of a graph G is the intersection graph on the family of cliques of G.

In a graph G, a vertex and an edge incident with it are said to **cover** each other. A set of vertices which cover all the edges is a vertex cover of G.

The **vertex covering number**α_0(G) of G is the minimum number of vertices in a vertex cover. A set of edges, which cover all the vertices, is an edge cover of G.

The **edge covering number**α_1(G) is the minimum number of edges in an edge cover. A set of S of vertices in G is independent if no two vertices in S are adjacent.

The **Independence number**β_0(G) of G is the maximum cardinality of an independent set of vertices. A set F of edges in G is independent if no two edges in F are adjacent.

The **edge independence number** (or the matching number) β_1(G) is the maximum cardinality of an independent set of edges. The maximum number of mutually adjacent vertices in G is the **clique number**(G) of G and edge clique number ω(G) of G is the maximum number of mutually adjacent edges in G.

1.2 Graph Coloring:

Graph coloring is one of the early areas of graph theory. Its origins may be traced back to 1852 when Augustus de Morgan in a letter to his friend William Hamilton asked if it is possible to color the regions of any map with four colors so that neighboring regions get different colors. This is the famous four color problem. The problem was first posed by Francis Guthrie, who observed that when coloring the countries of an administrative map of England only four colors were necessary in order to ensure that neighboring counties were given different colors. He asked if this was the case for every map and put the question to his brother Frederick, who was then a mathematics undergraduate in Cambridge. Frederick in turn informed his teacher Augustus de Morgan about the problem. In 1878 the four color problem was brought to the attention of the scientific community when Arthur Cayley presented it to the London Mathematical Society. It was proved that five colors are always sufficient, but despite heavy efforts it was not until 1977 that a generally accepted solution of the four color problem was published.

Graph coloring is a major sub-topic of graph theory with many useful applications as well as many unsolved problems. There are two types of graph colorings we will consider.

Vertex-Colorings and Edge-Colorings:

Given a set C called the set of colors (these could be numbers, letters, names, whatever), a function which assigns a value in C to each vertex of a graph is called a vertex-coloring. A proper vertex-coloring never assigns adjacent vertices the same color. Similarly, a function which assigns a value from a set of colors C to each edge in a graph is called an edge-coloring. A proper edge-coloring never assigns adjacent edges the same color.

In its simplest form **Graph Coloring**, is a way of coloring the vertices of a graph such that no two adjacent vertices share the same color, called a **vertex coloring**. Similarly, an **edge coloring** assigns a color to each edge so that no two incident edges share the same color, and a **face coloring** of a planar graph assigns a color to each face or region so that no two faces that share a boundary share the same color.

K-Coloring: A coloring of a graph using a set of k colors is called a k-coloring. A graph which has a k-coloring is said to be k-colorable.

The four-color theorem is equivalent to the statement that all planar graphs are 4-colorable. Note that a graph which is k-colorable might be colorable with fewer than k colors. It is often desirable to minimize the number of colors, i.e. find the smallest k.

Chromatic Number: The chromatic number of a graph G is the least k for which a k-coloring of G exists.

Other types of colorings:

Not only can the idea of vertex coloring be extended to edges, but also be added with different conditions to form new structures and problems.

Edge coloring:	Edges are colored.
List coloring:	Each vertex chooses from a list of colors.
List edge-coloring:	Each edge chooses from a list of colors.
Complete coloring:	Every pair of colors appears on at least one edge.
Acyclic coloring:	Every 2-chromatic subgraph is acyclic.
Strong coloring:	Every color appears in every partition of equal size exactly once.
Strong-edge coloring:	Edges are colored such that each color class induces a matching (equivalent to coloring the square of the line graph).
On-line coloring:	The instance of the problem is not given in advance and its successive parts become known over time.
Equitable coloring:	The sizes of color classes differ by at most one.
Total coloring:	Vertices and edges are colored.
Oriented coloring:	Takes into account orientation of edges of the graph.

Section II: Vertex Colorings and Upper Bounds:

When used without any qualification, a **coloring** is always assumed to be a vertex coloring, namely an assignment of colors to the vertices of the graph. Again, when used without any qualification, a coloring is nearly always assumed to be **proper**, meaning no two vertices are assigned the same color. Here, "adjacent" means sharing the same. A coloring using at most k colors is called a (proper) **k-coloring** and is equivalent to the problem of partitioning the vertex set into k or fewer.

Vertex coloring is the starting point of the subject, and other coloring problems can be transformed into a vertex version. The convention of using colors comes from graph drawings of graph colorings, where each node or edge is literally colored to indicate its mapping. In computer representations it's more typical to use nonnegative integers, and in general any mapping from the graph objects into a finite set can be used.

Chromatic Number: The least number of colors needed to color the graph is called its chromatic number. It is denoted by the symbol $\psi(G)$, where G is a graph. For example the chromatic number of a K_n of n vertices (a graph with an edge between every two vertices i.e., a Complete graph with n vertices), is $\psi(K_n) = n$. A graph that can be assigned(proper) k-coloring is k-colorable and it is k-chromatic if its chromatic number is exactly k.

Properties of Chromatic Numbers:

1. A graph, which is totally disconnected, has isolated vertices. No two vertices are adjacent. Therefore the graph is 1-chromatic.
2. A graph containing one edge or more edges is at atleast 2-chromatic.
3. A cycle with 3 vertices is 3-chromatic.
4. A graph which is a cycle of 2n points is 2-chromatic.
5. A complete graph with p-vertices is p-chromatic
6. A graph which is a cycle of 2n + 1 points is 3-chromatic.
7. If W_n is a wheel, having one vertex at the centre and n-1 vertices along the circumference, $\psi(W_n) = 4$ if n is even and $\psi(W_n) = 3$ if n is odd integer.

Color Class:

A color class is a set of vertices of a graph which are having the same color when a coloring is done to a graph.

Chromatic Polynomials and Chromatic Partitioning:

The **chromatic polynomial** counts the number of ways a graph can be colored using no more than a given number of colors.

In general, a given graph G of n vertices can be properly colored in many different ways using a sufficiently large number of colors. This property of a graph is expressed elegantly by means of a polynomial. This polynomial is called the **chromatic polynomial** of G and is defined as follows:

The value of the chromatic polynomial $P_n(\lambda)$ of a graph with n vertices gives the number of ways of properly coloring the graph, using λ or fewer colors.

Let $\mathbf{c_i}$ be the different ways of properly coloring G using exactly i different Colors. Since i colors can be chosen out of $\boldsymbol{\lambda}$ colors in

$\binom{\lambda}{i}$ or ${}^{\lambda}\mathbf{C_i}$ different ways,

i.e., there are $\binom{\lambda}{i}$ different ways of properly coloring G using exactly i colors out of λ colors.

Since i can be any positive integer from 1 to n(it is not possible to use more than n colors on n vertices), the chromatic polynomial is a sum of these terms; that is,

$$P_n(\lambda) = \sum c_i \binom{\lambda}{i}; \text{ i from 1 to n}$$

$$= c_1\frac{\lambda}{1!} + c_2\frac{\lambda(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + \dots + cn\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)\dots\dots(\lambda-n+1)}{n!}$$

Each c_i has to be evaluated individually for the given graph. For example, graph with even one edge requires at least two colors for proper coloring, and therefore,

$$c_1=0.$$

A graph with n vertices and using n different colors can be properly colored in n! Ways; that is,

$$c_n= n!$$

As an illustration, let us find the chromatic polynomial of the graph given in figure.

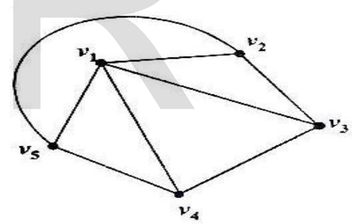

Fig.1:

$$P_5(\lambda)=c_1\frac{\lambda}{1!} + c_2\frac{(\lambda-1)}{2!} + c_3\frac{\lambda(\lambda-1)(\lambda-2)}{3!} + c_4\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3)}{4!} + c_5\frac{\lambda(\lambda-1)(\lambda-2)(\lambda-3).(\lambda-4)}{5!}$$

Since the graph in figure has a triangle, it will require at least three different colors for proper coloring. Therefore, c1=0 c_2= 0 and c5= 5! , Moreover, to evaluate c_3, suppose that we have three colors x, y, and z.

These three colors can be assigned properly to vertices v_1, v_2, and v_3 in 3! =6 different ways. Having done that, we have no more choices left, because vertex $v_5$must have the same color as v_3 and v_4 must have the same color as v_2. Therefore, c_3=6.

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Similarly, with four colors, v_1, v_2, and v_3 can be properly colored in 4.6=24 different ways. The fourth color can be assigned to v_4 or v_5, thus providing two choices. The fifth vertex provides no additional choice. Therefore,

$$c_4 = 24 .2 = 48.$$

Substituting these coefficients in $P_5(\lambda)$, we get, for the graph in figure.

$$P_5(\lambda)=\lambda(\lambda-1)(\lambda-2)+2\lambda(\lambda-1)(\lambda-2)(\lambda-3)+$$

$$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

$$=\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$$

The presence of factors $\lambda-1$ and $\lambda-2$ indicates, that G is atleast 3- chromatic.

Theorems and Proofs:

Theorem 1:

A graph of n vertices is complete graph if and only if its chromatic polynomial is,

$$P_n(\lambda)=\lambda(\lambda-1)(\lambda-2)\ldots\ldots\ldots(\lambda-n+1)$$

Proof: With λ colors, there are λ different ways of coloring any selected vertex of a graph. A second vertex can be colored properly in exactly λ -1 ways, the third in λ -2 ways, the fourth in λ -3 ways…, and the nth λ -n+1 ways if and only if every vertex is adjacent to every other. That is, if and only if the graph is complete.

Theorem 2:

Let aand b be two nonadjacent vertices in graph G. Let G' be a graph obtained by adding edge between a and b. Let G" be a simple graph obtained from G by fusing vertices a and b together and replacing sets of parallel edges with single edges. Then $P_n(\lambda)$ of G = $P_n(\lambda)$ of G' + $P_{n-1}(\lambda)$ of G"

Proof: The number of ways of properly coloring G can be grouped into two cases, one such that vertices a and b are of the same color and the other such that a and b are of different colors. Since the number of ways of properly coloring G such that a and b have different colors = numbers of ways of properly coloring G' and number of ways of properly coloring G such that a and b have the same color.

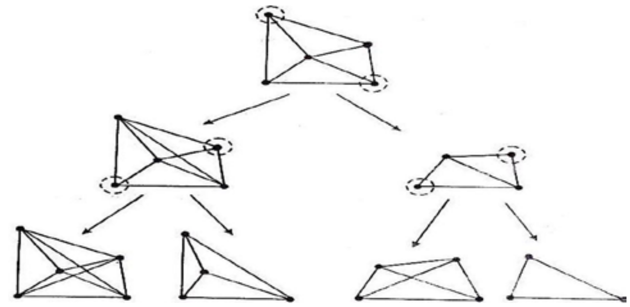

Fig.2:

$P_n(\lambda)$ of G' + $P_{n-1}(\lambda)$ of G"

$$=\lambda(\lambda-1)(\lambda-2)+2\lambda(\lambda-1)(\lambda-2)(\lambda-3)+$$

$$\lambda(\lambda-1)(\lambda-2)(\lambda-3)(\lambda-4)$$

$$=\lambda(\lambda-1)(\lambda-2)(\lambda^2-5\lambda+7)$$

= number of ways of properly coloring G

= $P_n(\lambda)$ of G

Theorem 3:

Berge and Ore theorem: In a k-chromatic graph, $\beta_o \geq \frac{p}{k}$ $\left[\text{Same as proving } \beta o \geq \frac{p}{\psi(G)}\right]$

Proof: $\psi(G)$ =k, because G is k – chromatic when we color with k colors, the vertices are partitioned into k – color classes, each color giving a class.

Let c_1, c_2, ...c_k be k color classes and $P_1 P_2$………..P_k be number of vertices in the color classes respectively

If β_o in the point independence number, $\beta_o$gives the maximum number of nonadjacent points and they have the same color.

Therefore β_o = maximum of (P_1, P_2,………..P_k)

$\beta_o \geq P_1$, $\beta_0 \geq P_2$..................... $\beta_o \geq P_i$, ...$\beta_o \geq P_k$

Where P_i is the maximum value of numbers (P_1, P_2,………..P_k)

$\beta_o + \beta_o$ ……………+ $\beta_o \geq P_1 + P_2$......... $+P_i+$……. P_k

$k\,\beta_o \geq P_1 + P_2 +$………..$P_k$

$\beta o \geq \frac{p}{k}$ Where p is the total number of points of the graph

i.e., $\beta o \geq \frac{p}{\psi(G)}$

Theorem 4:

Harary and Gaddum theorem: $\psi(G) \le p + 1 - \beta_o$ where p is the number of points in G.

Proof: Let S be the maximal independent set containing β_o points. Maximum number of points in the set G – S is $p - \beta_o$ ∴Maximum number of colors that could be used for coloring the set G – S is $P - \beta_o$. Minimum colors are used for coloring, therefore $\psi(G) - 1 \le P - \beta_o$

$$\psi(G) \le P - \beta_o + 1$$

Note: By theorem: $\beta_o \ge \frac{p}{\psi(G)}$

i.e $\psi(G) \ge \frac{p}{\beta o}$

By Theorem $\psi(G) \le p - \beta_o + 1$

Combining the two inequalities,

$$\frac{p}{\beta o} \le \psi(G) \le p - \beta_o + 1 .$$

Definition:

A k – coloring of G is a labeling f: V (G) → {1……k}. The labels are colors; A k coloring f is proper if x↔y implies f(x) ≠ f(y). A graph G is k – colorable if it has a proper k – coloring. The chromatic number ψ(G) is the minimum k such that, G is k – colorable, i.e., if ψ(G) = k , then G is k – chromatic but if ψ(H) < k for every proper sub graph H of G, then G is color critical or k – critical.

Example: In a proper coloring each color class is an independent set, therefore, G is k – colorable if and only if G is k – partite. Below we illustrate optimal colorings of the 5 – cycle and the Petersen graph, which have chromatic number 3.

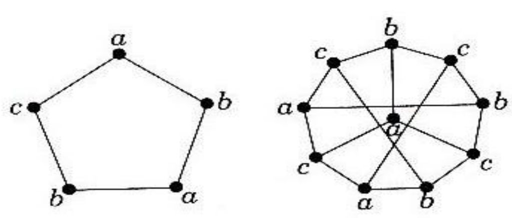

Fig.3:

Definition: The Cartesian product of graphs G and H written G x H, is the graph with vertex set V (G) x V (H) specified by putting (u , v) adjacent to (u', v') if and only if **(1)** u = u' and v v'∈ E (H), or **(2)** v = v' and u u' ∈ E (G).

Example: Cartesian products: The operation is symmetric: G x H ≅ H x G. For example, c_3x $k_2$appears below, and $Q_k = Q_{k-1}$ x k_2 if $k \ge 1$. In general, the edges of G x H can be partitioned into a copy of H for each vertex of G and a copy of G for each vertex of H.

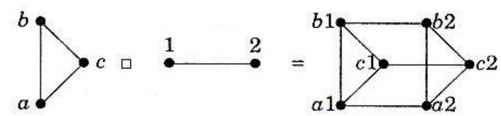

Fig.4:

UPPER BOUNDS:

Most upper bounds on ψ(G) come from coloring algorithms. The bound $\psi(G) \le n(G)$ where n(G) is the order of G.Uses nothing about the structure of G. We can improve the bound by coloring vertices successively using the "least available " color.

Algorithm: (Greedy coloring). The greedy coloring with respect to a vertex ordering $v_{1,}$..............v_n of V (G) is obtained by coloring vertices in the order $v_{1,}$..............v_n, assigning to v_i the smallest-indexed color not already used its lower-indexed neighbors.

PROPOSITION: $\psi(G) \le \Delta(G) + 1$

Proof: In a vertex ordering, each vertex has at most ψ(G) earlier neighbors, so the greedy coloring cannot be forced to use more than Δ (G) + 1 colors. This proves constructively that $\psi(G) \le \Delta(G) + 1$.

The bound $\psi(G) \le \Delta\psi(G) + 1$ results from every vertex ordering. By choosing the ordering carefully, we may obtain a better bound indeed, every graph G has a vertex ordering on which the greedy algorithm uses only ψ(G) colors.

Lemma: If H is a k – critical graph, then

$\partial(H) \ge k - 1$

Proof: suppose x is a vertex of H. Because H is k – critical, H – x is k – 1 colorable. If $d_H(x) < k-1$, then the k-1 colors used on H – x do not all appear on N(x), and we can assign a missing one to x to extend the coloring to H. This contradicts our hypothesis that H has no proper k – 1- coloring. Hence every vertex of H has degree at least k-1.

Theorem 5:

Brook's theorem [1941]: If a connected graph G is neither an odd cycle nor a complete graph,

Then **ψ**(G) ≤ Δ (G).

Proof: Suppose G is connected but it is not a clique or an odd cycle, and let k = Δ (G). We may assume that k ≥ 3, since G is a clique when k = 1, and G is an odd cycle or is bipartite when k = 2.

If G is not k-regular, choose v_n so that $d(v_n) < k$. Since G is connected, we can grow a spanning tree of G from v_n, assigning indices in decreasing order as we reach vertices. Each vertex other than v_n in the resulting ordering $v_1 \ldots v_n$ has higher indexed neighbor along the path to v_n in the tree. Hence each vertex has at most k – 1 lower indexed neighbors, and the greedy coloring uses at most k colors.

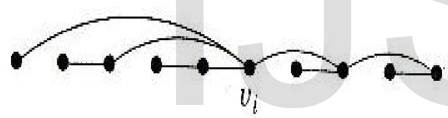

Fig.5:

In the other case, G is regular. If G has a cut–vertex x, let G' be a component of G-x together with its edges to x. The degree of x in G' is less than k, and we obtain a proper k-coloring of G' as above.

By permuting the names of colors in each such sub graph, we can make the colorings, agree on x to complete a proper k – coloring of G.

We may thus assume that G is 2- connected. Suppose G has an induced 3-vertex path, with vertices we call v_1, v_n, v_2 in order, such that G – {v_1, v2} is connected. We can then number the vertices of a spanning tree of G – {v1, v_2} using 3,….,n such that labels increase along paths to the root v_n. As before, each vertex before n as almost k-1 lower indexed neighbors. The greedy coloring also uses atmost k-1 colors on neighbors of V_n , since V_1 & V_2 receive the same color.

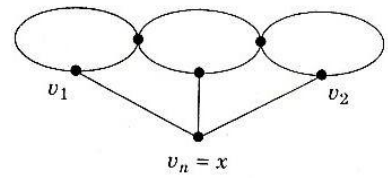

Fig.6:

Hence it enough to show that every 2-connected k-regular graph with k≥3 has three such vertices. Choose vertex x .If K (G-x) ≥ 2, let v_1 be x and let V_2 be a vertex with distance two from x, which exists because G is regular and not a clique. If K (G-x) = 1, then x has a neighbor in every block of G-x (since G has no cut-vertex). Neighbors V_1, V_2 of x in two such blocks are non-adjacent. Furthermore, G-{x, v_1, v_2} is connected, since blocks have no cut-vertices. Now k≥3 implies that G – {v_1, v_2} also is connected, and we let V_n= x . Hence the proof.

Section III: Edge Coloring

History of Edge Coloring: The edge-coloring problem is to color all edges of a given graph with the minimum number of colors so that no two adjacent edges are assigned the same color. In this chapter, we historically review the edge-coloring problem which was appeared in 1880 in relation with the four-color problem. The problem is that every map could be colored with four colors so that any neighboring countries have different colors. It took more than 100 years to prove the problem affirmatively in 1976 with the help of computers. The first paper dealing wit h the edge-coloring problem was written by Tait in 1880. In this paper Tait proved that if the four-color conjecture is true, then the edges of every 3-connected planar graph can be properly colored using only three colors. Several years later, in 1891 Petersen pointed out that there are 3-connected, cubic graphs which are not 3 colorable. The minimum number of colors needed to color edges of G is called the chromatic index $\chi^0(G)$ of G. Obviously $\chi^0(G) \geq \Delta(G)$, since all edges incident to the same vertex must be assigned different colors. In 1916, Konig has proved his famous theorem which states that every bipartite graph can be edge-colored with exactly Δ(G) colors, that is $\chi^0(G) = \Delta(G)$. In 1949, Shannon proved that every graph can be edge-colored with at most $\frac{3\Delta(G)}{2}$ colors, that is $\chi'(G) \leq \frac{3\Delta(G)}{2}$. In 1964, Vizing proved that $\chi^0(G) \leq \Delta(G) + 1$ for every simple graph.

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Definition: An edge coloring of a graph G is a function f :E(G) → C, where C is a set of distinct colors. For any positive integer k, a k-edge coloring is an edge coloring that uses exactly k different colors. A proper edge coloring of a graph is an edge coloring such that no two adjacent edges are assigned the same color. Thus a proper edge coloring f of G is a function

f: E(G)→ C such that f(e) =6 f(e^0) whenever edges e and e^0 are adjacent in G.

Definition: The chromatic index of a graph G, denoted

$\chi^0(G)$, is the minimum number of different colors required for a proper edge coloring of G. G is k-edge-chromatic if $\chi^0(G) = k$.

Theorem 6: For any graph G,$\Delta(G) \leq \chi^0(G) \leq 2\Delta(G) - 1$

Proof: An obvious lower bound for $\chi^0(G)$ is the maximum degree Δ(G) of any vertex in G. This is of course, because the edges incident one vertex must be differently colored. It follows that $\Delta(G) \leq \chi^0(G)$. The Upper bound can be found by using adjacency of edges Each edge is adjacent to at most Δ (G) − 1, other
edges at each of its endpoints. Thus,

$1 + (\Delta(G) - 1) + (\Delta(G) - 1) = 2\Delta(G) - 1$

Colors will always sufficient for a proper edge coloring of G.

Definition: The set of all edges receiving the same color in an edge coloring of G is called a color class. Alternatively a k-edge coloring can be thought of as a partition $(E_1,E_2,...,E_k)$ of E(G), where E_i denotes the (possibly empty) subset of E(G) assigned color i. If a coloring $\xi = (E_1,E_2,\cdots,E_k)$ is proper, then each $E_i$is a matching. Therefore $\chi^0(G)$ may be regarded as the smallest number of matchings into which the edge set of G can be partitioned. This interpretation of $\chi^0(G)$ will be helpful in the proof of certain useful results.

Theorem 7: Let G be a graph with m edges and let $m^*(G)$ be the size of a maximum matching. Then,

$$\chi'(G) \geq \lceil \frac{m}{m^*(G)} \rceil$$

Proof: . Consider coloring of the edges with using $q = \chi^0(G)$ colors $\alpha_1,\alpha_2,\cdots,\alpha_q$ and let E_i denotes the set of edges with color α_i. We have $m = |E_1| + |E_2| + \cdots + |E_q| \leq qm^*(G)$

Hence,$q \geq \frac{m}{m^*(G)}$and $\chi'(G) \geq \lceil \frac{m}{m^*(G)} \rceil$.

Proposition: Path Graphs:$\chi^0(P_n) = 2$, for $n \geq 3$.
Proposition: Cycle Graphs:

$$\chi'(C_n) = \begin{cases} 2 \text{ , if n is even; if n is odd.} \\ 3. \end{cases}$$

Proposition: Trees: $\chi^0(T) = \Delta(T)$, for any tree T.

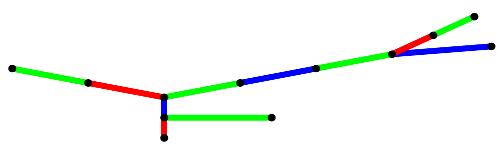

Fig.7:

Fig. 7: A proper edge 3-coloring of a tree.

Proposition: Wheel Graphs: $\chi^0(W_n) = n - 1$, for $n \geq 4$.

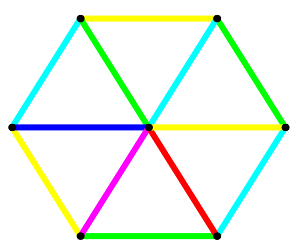

Fig.8: A proper edge 6-coloring of a wheel.

A 3-regular graph is also called a cubic graph. The best known cubic graph is the Petersen Graph (see Figure). The **Petersen Graph** is 3- regular with chromatic index 4. It is also not Hamiltonian. We will see now that these properties are connected.

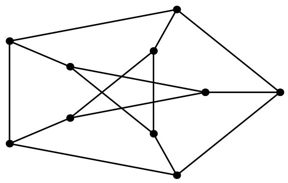

Fig.9: Petersen Graph

Theorem 8:Let G be a 3-regular graph with chromatic index 4. Then G is not Hamiltonian.

Proof: Since G is 3-regular then it must have an even number of vertices. Suppose G is Hamiltonian, then any Hamiltonian cycle of G is even, so we can color its edges properly with 2 colors, say red and blue. Now each vertex is incident with 1 red edge, 1 blue edge and 1 uncolored edge. The uncolored edges form a 1- factor of G, so we can color all of them with the same color, say green. Thus, G must be 3-edge-colorable, which is impossible. Therefore, G cannot be Hamiltonian.

Theorem 10: Vizing's Theorem:

Definition: Let G be a graph, and let f be a proper edge k-coloring of a subset S of the edges of G. Then f is blocked if for each uncolored edge e, every color has already been assigned to the edges that are adjacent to e. Thus, f cannot be extended to any edge outside
Subset S.

(Vizing's Theorem):

Let G be a simple graph. Then there exists a proper edge coloring of G that uses at most $\Delta(G) + 1$ colors. If G is a regular graph containing a cut-vertex, then G is of Class 2.

Proof: If G is of odd order, then the result follows from above stated Corollary. If G is of even order, let G=H∪K, where H ∩K = {v}. We may assume that H has odd order (say k), and that every vertex of H has degree $\Delta(G)$, except for v whose degree in H is less than $\Delta(G)$. It follows that the number of edges of H is:

$$m(H) = \frac{1}{2}[(k-1)\Delta(G) + deg_H(v)] > \Delta(G)\lfloor \frac{1}{2}k$$

And the result follows from Theorem:

Vizing Adjacency Lemma:

A graph G with atleast two edges is minimal with respect to chromatic index if $\chi^0(G - e) = \chi^0(G) - 1$ for every edge e of G. Since isolated vertices have no effect on edge colorings, it is natural to rule out isolated vertices when considering such minimal graphs. Therefore, the added hypothesis is that a minimal graph G is connected is equivalent to the assumption that G has no isolated vertices.

Two of the most useful results dealing with these minimal graphs are also results of Vizing, which are presented without proof.

Theorem 11: Let G be a connected graph of Class 2 that is minimal with respect to chromatic index. Then every vertex of G is adjacent to at least two vertices of degree $\Delta(G)$. In particular, G contains at least three Vertices of degree $\Delta(G)$.

Theorem 12: Let G be a connected graph of Class 2 that is minimal with respect to chromatic index. If u and v are adjacent vertices with deg(u) = k, then v is adjacent to at least $\Delta(G) - k + 1$ vertices of degree $\Delta(G)$.

Edge Colorings of Planar Graphs:

Definition: A planar graph is a graph which can be embedded in the plane so that no two edges intersect geometrically except at a vertex to which they are both incident.

Now let us consider edge colorings of planar graphs here. Our main problem remains to determine which planar graphs are of Class 1 and which are of Class 2.

Proposition: If G is a planar graph whose maximum degree is at most 5, then G can lie in either Class 1 or Class 2. It is easy to find planar graphs G of Class 1 for which $\Delta(G) = d$ for each $d \geq 2$ since all-star graphs are planar and of Class 1. There exist planar graphs G of Class 2 with $\Delta(G) = d$ for d = 2,3,4,5. For d = 2, the graph K_3 has the desired properties. It is not known whether there exists planar graphs of Class 2 having maximum degree 6 or 7; however Vizing proved that if G is planar and $\Delta(G) \geq 8$, then G must be of Class 1. We shall prove a similar, but weaker, result which may be found in his earlier paper.

Theorem 13: If G is a planar graph with $\Delta(G) \geq 10$, then G is of Class 1.

Proof: Suppose that the theorem is not true, and if suppose G is a planar graph of Class 2 with $\Delta(G) \geq 10$. Without loss of generality, that G is minimal with respect to chromatic index. Since G is planar, there must be at least one vertex in G whose degree is at most 5. Let S denote the set of all such vertices. Define H = G − S. Since H is planar, H contains a vertex w such that $deg_H(w) \leq 5$. Because $deg_G(w) > 5$, the vertex w is adjacent to vertices of S. Let v ∈S such that wv∈E(G), and let $deg_G(v) = k \leq 5$. Then by above stated Theorem, w is adjacent to at least d − k + 1 vertices of degree d, but d − k + 1 ≥ 6 so that w is adjacent to at least six vertices of degree d. Since d ≥ 10, w is adjacent to at least six vertices of H, contradicting the fact that $deg_H(w) \leq 5$.

As we mentioned above, this result can be improved to show that every planar graph with Δ(G) ≥ 8 is of Class 1. However the problem of determining what happens when the maximum degree is either 6 or 7 remains open.

Planar Graph Conjecture: Every planar graph with maximum degree 6 or 7 is of Class 1.

Enumerative Aspects

Sometimes we can shed light on a difficult problem by considering a more general problem. We know no good algorithm to compute the minimum k such that G has a proper k-coloring, but we can define ψ(G;k) to be the number of proper k-colorings of G. Knowing ψ(G;k) for all k would permit finding the minimum k where the value is positive, which is the Ψ(G).

Kirchhoff [1912] introduced this function as a possible way to attack the Four Color Problem.

Counting Proper Colorings

Definition: The function ψ(G ; k) counts the mappings f: V(G)→[k] that properly color G from the set [k] = {1,2,………..,k}. In this definition, the k colors need not all be used, and permuting the colors used produces a different coloring.

Example: When coloring the vertices of an independent set, we can independently choose one of the k colors at each vertex. Each of the k^n functions to [k] is a proper coloring, and hence $\psi(K_n; k) = k^n$.

Although K_3 has only one partition into three independent sets and none into four, we have $\psi(K_3;3) = 6$ and $\psi(K_3;4) = 24$. If we color $V(K_n)$ in some order, the colors chosen earlier cannot be used on the *i*th vertex, but there remain k-*i*+1 choices available for the *i*th vertex no matter how the earlier colors were chosen. Hence,

$\psi(K_n;k) = k(k-1)\ldots\ldots(k-n+1)$.

We obtain the same count by choosing *n* distinct colors and then multiplying by *n*! to count the ways each such choice can be assigned to the vertices. The value of the formula is 0 if k< n, as it should be since K_n is
K-chromatic. If we choose some vertex of a tree as a root, we can color it in k ways. If we grow the tree from the root, along with a coloring, at every stage only the color of the parent is forbidden, and we have k-1 choices for the color of the new vertex. Furthermore, by deleting a leaf, we can see inductively that every proper k-coloring arises in this way. Hence $\psi(T ; k) = k(k-1)^{n-1}$ for every *n*-vertex tree.

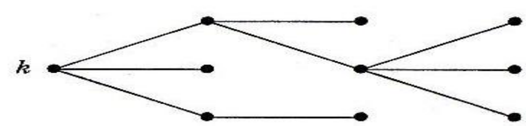

Fig.10:

The answers are polynomials in k of degree n(G). This holds for every graph, and hence ψ(G ; k) is called the chromatic polynomial of G.

Conclusion:

Graph coloring is still a very active field of research, I have studied and presented some important theorems on vertex coloring, edge colorings in this paper. We know no good algorithm to compute the minimum k such that G has a proper k-coloring, but from known theorems and proofs we can define ψ(G;k) to be the number of proper k-colorings of G. Knowing ψ(G;k) for all k would permit finding the minimum k where the value is positive, which is the Ψ(G).

Index of Symbols:	
G	Graph
V	Set of vertices
E	Set of edges
G-e	Deletion of edge
G-V	Deletion of vertex
[n]	{1, 2,…………., n}
G+H	Disjoint Union of graph
GxH	Cartesian product of sets
$\binom{n}{k}$	Binomial Coefficient
P_n	Path with n vertices
C_n	Cycle with n vertices
C(G)	Circumference
$d_{1,\ldots\ldots,}d_n$	Degree Sequence
$d(v)$, $d_G(v)$	Degree of V in G
$\Delta(G)$	Maximum Degree
$\delta(G)$	Minimum Degree
$d(u,v)$	Distance from U to V
$P_n(\lambda)$	Chromatic Polynomial
$\psi(G)$	Chromatic Number
K(G)	Clique graph
K_n	Complete graph
$K(n)$	Clique with *n* vertices
$K_{m,n}$	Complete bipartite graph
W_n	Wheel with n vertices
n(G)	Order (number of vertices)
T	Tree, tournament
α_o (G)	Vertex covering Number
α_1 (G)	Edge covering Number
β_o (G)	Independence number
β_1 (G)	Edge independence number
$W(G)$	Clique Number
$\psi'(G)$	Edge Chromatic Number
$\chi^0(G)$	Chromatic Index
$\psi(G ; k)$	Number of proper k-coloring of G

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